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Stable inversion-based robust tracking control in DC-DC nonlinear switched converters

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Abstract— **This article studies the robust tracking control problem in nonminimum phase DC-DC nonlinear switched power converters. The proposed procedure introduces a stable inversion-based iterative technique that, taking advantage of Banach's fixed point theorem, provides closed-form analytic expressions uniformly convergent to the exact solution of the corresponding inverse problem. Then, piecewise constant load disturbances belonging to an a priori known compact set are successfully rejected by means of dynamic compensation. Simulation results validate the proposal.**

I. INTRODUCTION

The possibility of using DC-to-DC switched mode power converters in AC voltage generation is specially interesting in the field of Uninterruptible Power Systems. As a consequence, robust step-down AC conversion with linear converters of the buck type is actually a fact [1]. However, efforts with nonlinear converters boost and buck-boost, which are candidates to carry out step-up conversion tasks, are handicapped by its well-known nonminimum phase character when direct control over the output voltage is exerted. This has usually led research to stable inversion control schemes that propound an indirect control of the output voltage via the input current [2].

Indeed, exact tracking of a known output reference $y = y_d$ in nonminimum phase, time-invariant systems by means of stable inversion was addressed in [3]. Since then, further studies have completed the method by relaxing original hypotheses, extending it to discrete-time systems and introducing new approaches (see [4] and the references therein). However, these general methods require backwards time numeric integration for the computation of a stable inverse, which is the main reason of the well known sensitivity of inversion-based exact-output tracking controllers to external perturbations and/or plant parameter uncertainties.

Hence, alternative approaches that contribute to robustness improvement for nonlinear power converters are scattered amongst literature. A common feature in these works is the obtention of closed-form analytic approximations of the unstable internal variable and its use in dynamic compensation schemes. Nevertheless, the validity of the proposals is shortened by different reasons: quality of the approximations [5], linearization procedures [6], lack of theoretical support [7] or hypotheses with a difficult checking process [8].

This article considers the output voltage tracking of periodic references in a class of nonminimum phase, DC-to-DC switched power converters using indirect control. The main contribution deals with the solution of the associated stable inversion problem and overcomes the lacks of [7], [8]. Namely, the method provides a uniformly convergent closed-form analytic iterative sequence of periodic approximations of a bounded periodic solution of the inverse problem that shows an explicit dependence on the system parameters. Then, a state feedback control law yields the nonminimum phase variable, i.e. the input current, to exactly track one of the approximate references, which has been selected a priori. This action results in the second state variable, i.e. the output voltage, exhibiting an asymptotically stable periodic response which is close to the original voltage reference. It is also proved that the use of the sequence of successive approximations of the indirect reference produces a sequence of periodic output voltages that converge uniformly towards the output command profile. Furthermore, the availability of closed-form analytic expressions for the indirect reference may yield robustness to piecewise constant load disturbances belonging to a known compact set by means of dynamic compensation once the disturbed/unknown parameters are measured or estimated. In turn, the steady-state output voltage tracking error should be reduced at will using a sufficiently high order iteration for the current command profile. The technique is supported on a number of assumptions that entail restrictions for which sufficient conditions on the output voltage reference candidates are derived. The methodology is exemplified and simulated on a boost converter.

Section II introduces the class of systems that are analyzed and the main problem to be solved. Section III is devoted to the stable inversion process. In Section IV the output tracking control problem is studied. The exemplification of the method on an ideal boost converter is detailed in Section V, and simulation results are in Section VI. Conclusions and suggestions for further research are collected in Section VII.

II. PROBLEM FORMULATION

The state-space averaged model of the DC-to-DC switched power converters boost and buck-boost is given by

$$L\frac{di_{\mathrm{L}}}{d\tau} = -v_{\mathrm{C}} + \mu v_{\mathrm{C}} + V_{\mathrm{g}}\left[1 + k(\mu - 1)\right] \quad (1)$$

$$C\frac{dv_{\mathrm{C}}}{d\tau} = i_{\mathrm{L}} - \frac{v_{\mathrm{C}}}{R} - \mu\, i_{\mathrm{L}}, \quad (2)$$

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where the inductor current i_L and the capacitor voltage v_C act as state variables and the control action $\mu = \mu(t)$ takes values in the interval $(0,1)$. Recall that the control action in the physical converter is actually carried out by means of a switch; hence, $\mu(t)$ is implemented through a PWM signal. The constant voltage source V_g, the inductance L and the capacitance C are considered well known parameters, while perturbations may affect the load resistance R. $k=0$ and $k=1$ model, respectively, the boost and the buck-boost converter.

An appropriate change of state variables and a time rescaling yield a dimensionless model with a minimum number of parameters that simplify a systematic analysis. Namely, using $x_1 = V_g^{-1}\left(LC^{-1}\right)^{1/2} i_L,\ x_2 = V_g^{-1} v_C,\ t = (LC)^{-1/2}\tau$, and setting $\lambda = R^{-1}\left(LC^{-1}\right)^{1/2},\ u = 1-\mu$, (1), (2) become

$$\dot{x}_1 = 1 - u(x_2 + k) \tag{3}$$

$$\dot{x}_2 = -\lambda x_2 + u x_1. \tag{4}$$

Since L, R and C take positive values, λ is always positive. Meanwhile, $u : [0,+\infty) \longrightarrow (0,1)$.

The elimination of u in (3),(4) yields the following differential relation between state variables:

$$x_1(1-\dot{x}_1) = (x_2+k)(\dot{x}_2 + \lambda x_2). \tag{5}$$

Let the control objective be the tracking of a smooth, T-periodic reference $x_{2d}(t)$ by the state variable x_2. The study of the zero dynamics reveals the nonminimum phase character of the system when $y = x_2$ is chosen as output, as well as the minimum phase character for $y = x_1$ (see [9], for example). This fact suggests the possibility of a stable inversion control approach that may indirectly achieve the control objective $x_2 = x_{2d}(t)$ in the steady state by forcing x_1 to track a suitable reference x_{1d}. Such a command profile, which should be bounded and, preferable, also T-periodic, arises from the corresponding inverse problem defined through (5):

$$x_1(1-\dot{x}_1) = \left[x_{2d}(t)+k\right]\left[\dot{x}_{2d}(t) + \lambda x_{2d}(t)\right]. \tag{6}$$

It follows from [9] that (6) shows a positive, $\mathscr{C}^1$, T-periodic solution $x_1(t) = x_{1d}(t) = \phi(t)$ whenever

$$g(t) = \left[x_{2d}(t)+k\right]\left[\dot{x}_{2d}(t) + \lambda x_{2d}(t)\right] \tag{7}$$

is also T-periodic, positive and smooth. However, this solution results to be unstable, in accordance to the nonminimum phase character of the system when $y = x_2$. An iterative procedure to find a closed-form analytic sequence of approximate solutions of (6) which are bounded, T-periodic, explicitly parameter-dependent and exhibit uniform convergence to ϕ is detailed in next Section.

III. STABLE INVERSION

Consider the Abel equation in the normal form

$$x\dot{x} = x - g(t). \tag{8}$$

The following result is proved in [9]:

Theorem 1: Let $g(t)$ be a T-periodic, smooth function and such that $g(t) > 0,\ \forall t \geq 0$. Then, equation (8) has one and only one T-periodic solution $\phi(t)$, which is positive and unstable.

Let us obtain an iterative sequence of successive approximations of the limit cycle of (8) by means of Banach's fixed-point theorem.

For, let $T \in \mathbb{R}^+$ and denote by $\mathscr{C}^n_{per}([0,T])$, $n = 0,1,\ldots,\infty$, the subset of elements of $\mathscr{C}^n([0,T])$ that allow a continuous and T-periodic extension in $\mathbb{R}$, i.e.

$$\mathscr{C}^n_{per}([0,T]) = \{x \in \mathscr{C}^n([0,T]);\ x(0) = x(T)\},$$

with $\mathscr{C}_{per}([0,T]) = \mathscr{C}^0_{per}([0,T])$. It is well known that $(\mathscr{C}_{per}([0,T]), \|\cdot\|)$, where $\|\cdot\|$ is the supremum norm, i.e.

$$\|x\| = \sup_{[0,T]}\{|x(t)|\},\ x \in \mathscr{C}([0,T]),$$

is a Banach space with respect to the metric induced by $\|\cdot\|$:

$$d(x,y) = \|x-y\|,\ x,y \in \mathscr{C}_{per}([0,T]).$$

Consider now $P_0 : \mathscr{C}_{per}([0,T]) \longrightarrow \mathbb{R}$, a projection operator that acts in the following way:

$$P_0 x = \frac{1}{T}\int_0^T x(t)dt,\ x \in \mathscr{C}_{per}([0,T]),$$

and let $\bar{X}$ denote the subset of $\mathscr{C}_{per}([0,T])$ that contains the elements with zero mean value:

$$\bar{X} = \{x \in \mathscr{C}_{per}([0,T]);\ \int_0^T x(t)dt = 0\}.$$

Then, it is immediate that any $x \in \mathscr{C}_{per}([0,T])$ can be uniquely decomposed as

$$x = x_0 + \bar{x}, \quad \text{with} \quad x_0 = P_0 x \quad \text{and} \quad \bar{x} \in \bar{X}. \tag{9}$$

Finally, $\bar{X}$ being closed by integration, for all $\bar{x} \in \bar{X}$ there exists a unique element $\hat{x} \in \bar{X}$ such that

$$\dot{\hat{x}} = \bar{x}. \tag{10}$$

Assumption A. Let $x_{2d}(t)$ and the corresponding function $g(t)$ defined in (7) be positive, $C^\infty_{per}([0,T])$ verifying

$$g_0 > \frac{T}{2} + \sqrt{2\|\hat{g}\|}. \tag{11}$$

Lemma 2: Let us define

$$\alpha = \frac{1}{g_0}\sqrt{\left(g_0 - \frac{T}{2}\right)^2 - 2\|\hat{g}\|},$$

$$L_- = g_0(1-\alpha) - \frac{T}{2},\quad L_a = a g_0 - \frac{T}{2}.$$

If Assumption A holds, then:
(i) $0 < \alpha \leq 1 - T(2g_0)^{-1}$.
(ii) $\forall a \in (1-\alpha, 1)$, $0 \leq L_- < L_a$.

Lemma 3: Let $\bar{x}, \hat{x} \in \bar{X}$ be such that $\dot{\hat{x}} = \bar{x}$. Then, $\|\hat{x}\| \leq \frac{T}{2}\|\bar{x}\|,\ \forall \bar{x} \in \bar{X}$

Proof: $\hat{x}$ being continuous and with zero mean value in $[0,T]$, it is straightforward that there exists, at least, $t_0 \in [0,T]$ such that $\hat{x}(t_0) = 0$. Therefore, assuming that both $\bar{x}(t)$ and $\hat{x}(t)$ are naturally extended to $[t_0, t_0+T]$, one has that

$$\int_{t_0}^{t_0+T} \hat{x}(t)dt = 0.$$

Moreover, the T-periodicity of $\hat{x}(t)$ ensures the existence of $c \in [t_0, t_0+T]$ such that $\|\hat{x}\| = |\hat{x}(c)|$; hence,

$$\|\hat{x}\| = |\hat{x}(c)| = \left| \int_{t_0}^{c} \bar{x}(t)dt \right|.$$

Furthermore, it is also straightforward that

$$\int_0^T \bar{x}(t)dt = 0 \Longrightarrow \int_{t_0}^{t_0+T} \bar{x}(t)dt = 0;$$

thus,

$$\int_{t_0}^{c} \bar{x}(t)dt = \int_{t_0+T}^{t_0} \bar{x}(t)dt + \int_{t_0}^{c} \bar{x}(t)dt = \int_{t_0+T}^{c} \bar{x}(t)dt,$$

this yielding

$$\|\hat{x}\| = \left| \int_{t_0}^{c} \bar{x}(t)dt \right| \leq (c-t_0)\,\|\bar{x}\|, \quad \text{and}$$

$$\|\hat{x}\| = \left| \int_{t_0+T}^{c} \bar{x}(t)dt \right| \leq (t_0+T-c)\,\|\bar{x}\|.$$

The result follows adding the two inequalities . ■

The main result of the section reads as follows.

Theorem 4: If Assumption A holds, then $\forall a \in (1-\alpha, 1)$ and $\forall L \in (L_-, L_a]$, there exist a closed, nonempty subset M_L of $\mathscr{C}_{per}([0,T])$ defined as

$$M_L = \{\bar{x} \in \bar{X}; \quad \|\bar{x}\| \leq L\}, \tag{12}$$

such that the sequence $\{x_n\}$=$\{g_0+\bar{x}_n\}$, obtained by means of the iterative procedure

$$\bar{x}_{n+1} = \bar{A}(\bar{x}_n) = \frac{1}{g_0}\left[\hat{x}_n - \hat{g} - \frac{1}{2}\left(\bar{x}_n^2 - P_0\bar{x}_n^2\right)\right],\ \bar{x}_0 \in M_L, \tag{13}$$

converges uniformly to the hyperbolic, T-periodic solution $\phi(t)$ of (8).

Proof: Follow Section 4 in [10] and replace Lemma 6 therein by the above stated Lemma 3. ■

Corollary 5: Let Assumption A hold and consider the Fourier expansion of $g(t)$:

$$g(t) = g_0 + \bar{g}(t) = g_0 + \sum_{k\geq 1} A_k \cos k\omega t + B_k \sin k\omega t. \tag{14}$$

Let the initial condition $\bar{x}_0 \in M_L$ in (13) be selected as a $\mathscr{C}^1([0,T])$ function, its Fourier expansion being

$$\bar{x}_0(t) = \sum_{k\geq 1} \alpha_{0k} \cos k\omega t + \beta_{0k} \sin k\omega t. \tag{15}$$

Then, $\forall n \geq 0$, the Fourier expansions of the successive approximations $x_n = g_0+\bar{x}_n$ obtained from (13) converge uniformly to the corresponding x_n and follow the recursive assignment $x_{n+1}(t) = g_0 + \sum_{k\geq 1} \alpha_{n+1,k} \cos k\omega t + \beta_{n+1,k} \sin k\omega t$, with

$$\begin{aligned}
\alpha_{n+1,k} &= \tfrac{B_k-\beta_{nk}}{kg_0 w} - \tfrac{1}{2g_0}\textstyle\sum_{j\geq 1}\left(\alpha_{n,k+j}\alpha_{nj} + \beta_{n,k+j}\beta_{nj}\right) + \\
&\quad + \tfrac{1}{4g_0}\textstyle\sum_{j=1}^{k-1}\left(\beta_{n,k-j}\beta_{nj} - \alpha_{n,k-j}\alpha_{nj}\right) \\
\beta_{n+1,k} &= \tfrac{\alpha_{nk}-A_k}{kg_0 w} + \tfrac{1}{2g_0}\textstyle\sum_{j\geq 1}\left(\alpha_{n,k+j}\beta_{n,j} - \alpha_{n,j}\beta_{n,k+j}\right) + \\
&\quad - \tfrac{1}{2g_0}\textstyle\sum_{j=1}^{k-1} \alpha_{n,k-j}\beta_{nj}.
\end{aligned}$$

Proof: The result follows from straightforward computation after taking into account Theorem 4, the operations involved in (13) and the convergence properties of Fourier series expansions (see, for example, [11]). ■

Remark 1: It is immediate from Corollary 5 that the Fourier coefficients of the successive approximations x_n depend explicitly on $\alpha_{0j}, \beta_{0j}, A_j, B_j$, i.e. $\alpha_{nk} = \alpha_{nk}(\alpha_{0j}, \beta_{0j}, A_j, B_j)$, $\beta_{nk} = \beta_{nk}(\alpha_{0j}, \beta_{0j}, A_j, B_j)$.

When the hypotheses of Corollary 5 are satisfied and Fourier expansions of $g(t)$ and $\bar{x}_0$ (see (14) and (15)) are finite, the coefficients α_{nk}, β_{nk} are not only numerically but analytically computable in closed form.

IV. OUTPUT TRACKING CONTROL

Throughout this Section, ϕ stands for the positive, T-periodic solution of (8), and $\{\phi_n\} = \{g_0 + \bar{\phi}_n\}$, with $\bar{\phi}_n$ obtained as indicated in Theorem 4, denotes a sequence of T-periodic approximations of ϕ.

Assume that system (3),(4) undergoes a state feedback control action $u(t)$ that forces a steady-state in which the state variable x_1 tracks a reference signal $x_{1d}(t)$. Namely,

$$u = \frac{1-\dot{x}_{1d} + \gamma(x_1 - x_{1d})}{x_2+k}, \quad \gamma \in \mathbb{R}^+, \tag{16}$$

which yields $x_1 = x_{1d}(t)$ in the steady state. Hence, (5) gives the internal dynamics equation that governs x_2:

$$(x_2+k)\,(\dot{x}_2 + \lambda x_2) = x_{1d}(1-\dot{x}_{1d}). \tag{17}$$

Assume that x_{1d} is selected as a certain approximation of $\phi(t)$, say $x_{1d} = \phi_n$. Therefore, (17) results in

$$(x_{2n}+k)\,(\dot{x}_{2n} + \lambda x_{2n}) = \phi_n(1-\dot{\phi}_n), \tag{18}$$

x_{2n} being the output corresponding to $x_1 = \phi_n$. However, as ϕ_n is an approximate solution of (8), $\forall n \geq 0$ there exists a T-periodic function $F\phi_n(t)$ such that

$$F\phi_n(t) = \phi_n(1-\dot{\phi}_n) - \phi(1-\dot{\phi}) = \phi_n(1-\dot{\phi}_n) - g(t), \tag{19}$$

with $g(t) = g(t, x_{2d}(t))$ (recall (7)). Therefore, (18) may be written as

$$(x_{2n}+k)\,(\dot{x}_{2n} + \lambda x_{2n}) = G_n(t), \tag{20}$$

where

$$G_n(t) = \phi_n(1-\dot{\phi}_n) = g(t) + F\phi_n(t). \tag{21}$$

Theorem 6: [8] Let $G_n(t) > 0$, $\forall t \geq 0$. Then, the ODE (20) has one and only one T-periodic solution $x_{2n}(t)$ in $\mathbb{R}^+$, which is asymptotically stable.

As a consequence of Theorem 6, in case that every element of the sequence $\{G_n\}$ is positive, (20) produces a sequence $\{x_{2n}\}$ of positive, T-periodic and asymptotically stable responses. Thus, let us establish sufficient conditions to ensure that a certain sequence $\{\phi_n\}$ yields $G_n > 0$, $\forall n \geq 0$.

Assumption B. Assumption A holds and the radius L of the set M_L defined in (12) also satisfies

$$L < \frac{g_0 - \|\bar{g}\|}{2}. \tag{22}$$

Furthermore, the sequence $\{\phi_n\}$ is such that $\|\dot{\phi}_0\| \leq D$, D being a positive real number verifying

$$\frac{\|\bar{g}\| + L}{g_0 - L} \leq D < 1. \tag{23}$$

Remark 2: The existence of such a positive D is ensured as follows: the selection of a radius L according to Theorem 4, together with Lemma 2, yields

$$g_0 - L \geq g_0 - L_a = (1-a)g_0 + \frac{T}{2} > 0. \tag{24}$$

Moreover, using (22) one gets: $0 < \|(\bar{g}\| + L)(g_0 - L)^{-1} < 1$.

Proposition 7: Let Assumption B hold. Then, the sequence $\{G_n\}$ defined in (21) verifies that $G_n(t) > 0$, $\forall n \geq 0$.

Proof: On the one hand, let $\bar{\phi}_0 \in M_L$; then, by Theorem 4, $\bar{\phi}_n \in M_L$, $\forall n \geq 0$. Consequently, $\forall n \geq 0$, (24) entails

$$\phi_n = g_0 + \bar{\phi}_n \geq g_0 - \|\bar{\phi}_n\| \geq g_0 - L > 0. \tag{25}$$

On the other hand one has that $\bar{\phi}_0 \in M_L$ and $\|\dot{\phi}_0\| = \|\dot{\bar{\phi}}_0\| \leq D < 1$ by hypothesis. In accordance to the induction principle, assume that $\|\dot{\phi}_n\| = \|\dot{\bar{\phi}}_n\| \leq D < 1$. Now, using (13) and (23), $\|\dot{\phi}_{n+1}\| = \leq g_0^{-1}(g_0 D - LD + LD) \leq D < 1$. Therefore, $1 - \dot{\phi}_n \geq 1 - \|\dot{\phi}_n\| \geq 1 - D > 0, \quad \forall n \geq 0$, and, finally, $G_n = \phi_n(1 - \dot{\phi}_n) > 0, \quad \forall n \geq 0$. ■

The uniform convergence of the output sequence $\{x_{2n}\}$ under approximate control is now ready to be stated.

Theorem 8: Let Assumption B hold. Then, the sequence $\{x_{2n}\}$ converges uniformly to the output reference $x_{2d}(t)$.

Proof: Follow Section 5 in [8]. ■

The applicability of the control procedure is restricted to the fulfillment of an obviously necessary condition: the steady-state control law $u_{ss}(t) = u(t, x_1 = x_{1d}, x_2 = x_{2ss})$ must lie in $(0,1)$, $\forall t \geq 0$, with x_{2ss} standing for the steady-state solution of (17). Hence, let us now proceed with a brief discussion of the special requirements of the steady-state behavior of system (3), (4) associated with the bounded character of the control gain u.

The isolation of u in system (3),(4) yields

$$u = \frac{1 - \dot{x}_1}{x_2 + k} = \frac{\dot{x}_2 + \lambda x_2}{x_1}; \tag{26}$$

as $u \in (0,1)$, the state trajectories (x_1, x_2) fulfilling (5) that do not lead to saturation of the control action are those satisfying

$$0 < \frac{1 - \dot{x}_1}{x_2 + k} < 1,\ \forall t \geq 0, \quad \text{or} \quad 0 < \frac{\dot{x}_2 + \lambda x_2}{x_1} < 1,\ \forall t \geq 0. \tag{27}$$

Moreover, straightforward calculation results in:

From now on, let $\{x_{2n}\}$ denote the sequence of output responses derived from the use of the elements of $\{\phi_n\}$ in the internal dynamics equation (20).

Proposition 9: Let Assumption B hold. Then,

(i) The unsaturated region of the state-space (27), corresponding to the steady-state of system (3),(4) when it undergoes exact tracking control, i.e. $x_1 = x_{1d}(t) = \phi(t)$, $x_2 = x_{2d}(t)$, is equivalently defined by

$$0 < 1 - \dot{\phi} < x_{2d} + k, \quad \text{or} \quad 0 < \dot{x}_{2d} + \lambda x_{2d} < \phi. \tag{28}$$

(ii) The unsaturated region of the state-space (27), corresponding to the steady-state of system (3),(4) when it undergoes approximate tracking control, i.e. $x_1 = x_{1d}(t) = \phi_n(t)$, $x_2 = x_{2d}(t) = x_{2n}(t)$, is equivalently defined by

$$0 < 1 - \dot{\phi}_n < x_{2n} + k, \quad \text{or} \quad 0 < \dot{x}_{2n} + \lambda x_{2n} < \phi_n. \tag{29}$$

Proof: Notice that if $x_1(t) > 0$ and $x_2(t) + k > 0$, $\forall t \geq 0$, the unsaturated region (27) is equivalently defined by $0 < 1 - \dot{x}_1 < x_2 + k$ or $0 < \dot{x}_2 + \lambda x_2 < x_1$. Then, (i) follows immediately after taking into account that $x_{2d}(t) > 0$, $\phi(t) > 0$, $\forall t \geq 0$, from Assumption B and Theorem 1. Equivalently, $\phi_n(t) > 0$ also follows from Assumption B (recall (25) in the proof of Proposition 7), while Proposition 7 and Theorem 6 ensure $x_{2n}(t) > 0$, $\forall t \geq 0$, this yielding (ii). ■

The reasonable demand of unsaturation of the control action in the steady-state is claimed straightforward:

Assumption C. Assumption B holds and the steady-state of system (3),(4) remains in the unsaturated region defined by (28) (resp. (29)) when it undergoes exact (resp. approximate) tracking control, i.e. $x_1 = x_{1d}(t) = \phi(t)$, $x_2 = x_{2d}(t)$ (resp. $x_1 = x_{1d}(t) = \phi_n(t)$, $x_2 = x_{2n}(t)$, $\forall n \geq 0$).

The Section closes with the derivation of restrictions involving the output reference candidate x_{2d} that may help in ensuring the fulfillment of Assumptions B and C.

Proposition 10: Let Assumption A hold. Then, it is a necessary condition for the fulfillment of Assumption B that

$$\frac{g_0 + \|\bar{g}\| - T}{2} < \sqrt{\left(g_0 - \frac{T}{2}\right)^2 - 2\|\hat{g}\|}. \tag{30}$$

Proof: It is immediate from Theorem 4 that the selected radius L of M_L is lower bounded by L_-. Hence, $L_- < 1/2(g_0 - \|\bar{g}\|)$ is a necessary condition for the fulfillment of Assumption B; (30) follows re-writing the later inequality using the definitions of α and L_- in Lemma 2. ■

Proposition 11: Let Assumption B hold. Then, it is sufficient for the fulfillment of Assumption C that:

$$\inf_{t\in[0,T]}\{g(t)\} \geq \|\dot{x}_{2d}(t) + \lambda x_{2d}(t)\| \tag{31}$$

$$g_0 - L > \lambda \frac{(1+D)^2}{1-D} \tag{32}$$

Proof: It is proved in [9] that (31) is a sufficient condition for the second inequality of (28) to be verified.

Let us now consider the first inequality of (29). Assumption B and Proposition 7 ensure that $1 - \dot{\phi}_n > 0$, $\forall n \in \mathbb{N}$; thus, $x_{2n} + k > 1 - \dot{\phi}_n$ is ensured if $\inf_{t\in[0,T]}\{x_{2n} + k\} > \|1 - \dot{\phi}_n\|$. Recall that x_{2n} is a $\mathscr{C}^1_{per}([0,T])$ that satisfies (18). Hence, there exists $t_m \in [0,T]$, with $\dot{x}_{2n}(t_m) = 0$, where $x_{2n}(t)$ attains minimum value, this yielding

$$\lambda x_{2n}(t_m)\left[x_{2n}(t_m) + k\right] = \phi_n(t_m)\left[1 - \dot{\phi}_n(t_m)\right].$$

Moreover, as $\lambda\left[x_{2n}(t_m) + k\right]^2 \geq \lambda x_{2n}(t_m)\left[x_{2n}(t_m) + k\right]$, one gets that $x_{2n}(t_m) + k \geq \sqrt{\lambda^{-1}\phi_n(t_m)\left[1 - \dot{\phi}_n(t_m)\right]}$. Then,

$$\inf_{t\in[0,T]}\left\{\sqrt{\frac{1}{\lambda}\phi_n(t)\left[1 - \dot{\phi}_n(t)\right]}\right\} > \|1 - \dot{\phi}_n\| \tag{33}$$

guarantees the claim. Furthermore, (33) is equivalent to

$$\inf_{t\in[0,T]}\left\{\phi_n(t)\left[1 - \dot{\phi}_n(t)\right]\right\} > \lambda\|\left(1 - \dot{\phi}_n\right)^2\|,$$

and, using Assumption B:

$$\inf_{t\in[0,T]}\left\{\phi_n(t)\left[1 - \dot{\phi}_n(t)\right]\right\} \geq (g_0 - L)(1 - D),$$

$$\lambda\|\left(1 - \dot{\phi}_n\right)^2\| \leq \lambda\|1 - \dot{\phi}_n\|^2 \leq (1+D)^2.$$

Therefore, (32) is a sufficient condition for the fulfillment the first inequality of (29). Finally, as Proposition 9 guarantees that, under Assumption B, the unsaturated regions to which Assumption C refers are equivalently defined by any of (28) or (29), the result follows. ■

V. ROBUST TRACKING OF SINUSOIDAL SIGNALS WITH A BOOST CONVERTER

The setting $k = 0$ in (3), (4) defines the boost converter. Let the control goal be the robust tracking of

$$x_{2d}(t) = A + B\sin\omega t, \quad A > B > 0, \tag{34}$$

by the output voltage, with λ affected by possible perturbations that may vary its value in the known compact set $\Lambda = [\lambda_-, \lambda_+] \subset \mathbb{R}^+$. The proposed procedure is based on dynamic compensation of disturbance effects through an on-line updating of the selected current reference $\phi_n(t) = \phi_n(t,\lambda)$ according to the instantaneous variation of λ, which is assumed to be estimated (using, for example, the algebraic estimator [12]) or measured. This fact implies that the study carried out in Section IV may be applicable in such a perturbed situation whenever Assumptions A, B and C are ensured for all $\lambda \in \Lambda$. Hence, the section is devoted to establish conditions for the fulfillment of Assumptions A, B and C when a boost converter is demanded to track the command profile (34) in the presence of load perturbations in a known compact set Λ.

Let us first define the auxiliary functions:

$$\bar{B}(\lambda) = B\sqrt{1+\left(\frac{\omega}{\lambda}\right)^2}, \quad K(\lambda) = A\sqrt{\bar{B}^2(\lambda)+3B^2} + \frac{B\bar{B}(\lambda)}{2},$$

where $\bar{B}_\pm = \bar{B}(\lambda_\pm)$, $K_\pm = K(\lambda_\pm)$, is adopted for simplicity.

Proposition 12: It is sufficient for the fulfillment of Assumption A in Λ that $A > \bar{B}_-$ and also that:

$$A^2 + \frac{B^2}{2} > \frac{T}{2\lambda_-} + \sqrt{\frac{T}{4\pi\lambda_-}\left(4K_- - B\bar{B}_-\right)}. \tag{35}$$

Proof: (34) yields $x_{2d} > 0$ and also $x_{2d}, g \in \mathscr{C}^\infty_{per}([0,T])$. Moreover, from (7):

$$g(t) = \lambda(A + B\sin\omega t)\left[A + \bar{B}\sin\left(\omega t + \arctan\frac{\omega}{\lambda}\right)\right].$$

Therefore, $g(t) \geq \lambda(A-B)(A-\bar{B})$ $\forall t \in [0,T]$ and, being $A > B > 0$ by hypothesis, $A > \bar{B}_-$ guarantees $g > 0$, $\forall \lambda \in \Lambda$. Furthermore, using (9), (10), one gets that $g_0 = \lambda\left(A^2 + B^2/2\right)$,

$$\|\bar{g}\| \leq \lambda K(\lambda), \quad \|\hat{g}\| \leq \frac{\lambda T}{8\pi}\left[4K(\lambda) - B\bar{B}(\lambda)\right], \tag{36}$$

where the second and third expressions are obtained using the triangle inequality and also that $\|a\sin\alpha + b\cos\alpha\| = \sqrt{a^2+b^2}$, $\forall (a,b)$. Hence, (35) entails (11), $\forall \lambda \in \Lambda$. ■

Proposition 13: Let $N = 4K_- - B\bar{B}_-$. It is necessary for the fulfillment of Assumption B in Λ that

$$A^2 + \frac{B^2}{2} > \frac{1}{3}\left[\frac{T}{\lambda_-} + K_- + \sqrt{\left(\frac{T}{\lambda_-}\right)^2 + 2K_-^2 + \frac{TN}{\pi\lambda_-}}\right] \tag{37}$$

Proof: Algebraic manipulation indicates that (30) is equivalent to $g_0 > 3^{-1}\left(T + \|\bar{g}\| + (T^2 + 2\|\bar{g}\|^2 + 8\|\hat{g}\|)^{1/2}\right)$. Using (36) it is immediate that (37) is sufficient for the fulfillment of (30), $\forall \lambda \in \Lambda$. According to Propositon 10, the result follows. ■

Proposition 14: Let Assumption B be satisfied. Then, it is sufficient for (31) to hold, $\forall \lambda \in \Lambda$, that

$$A - B > \frac{A + \bar{B}_+}{A - \bar{B}_+} \tag{38}$$

Proof: See Chapter 4.6 in [13]. ■

VI. SIMULATION RESULTS

The technique has been tested in a boost converter with $V_g = 50V$, $L = 0.018H$, $C = 0.00022F$ and $R_N = 10\Omega$. The output voltage reference profile has been set to:

$$v_C(\tau) = 210 + 50\sin(2\pi\nu\tau),$$

with $\nu = 50Hz$. At a certain time instant, the load resistance is assumed to undergo an additive perturbation of a 50% of the nominal value R_N, thus growing up to $R_P = 15\Omega$.The corresponding values in normalized variables are $\Lambda = [\lambda_-, \lambda_+] = [0.6030, 0.9045]$ and $x_{2d}(t) = 4.2 + \sin\omega t$, where $\omega = 0.6252$.

With these settings it is possible to verify Assumptions A, B and C, $\forall \lambda \in \Lambda$:

- Restrictions $A > \bar{B}_-$ and (35) are fulfilled; according to Proposition 12, Assumption A is satisfied.
- The necessary condition (37) holds. The contractive constant a is to be selected in $(0.5359, 1)$; once a is fixed, the possible radius L of the set M_L defined in (12) belong to $(L_-, L_a]$, with $L_- \leq 0.8371$ and $L_a \geq 10.9288a - 5.0252$. Let $a = 0.9$ and $L = 1$. This entails $D \in (0.724, 1]$, so we choose $D = 0.8$. Hence, both (22) and (23) hold $\forall \lambda \in \Lambda$ and consequently, so does Assumption B.
- Inequality (38) is verified. As the selected values for L and D in the former item also guarantee (32), $\forall \lambda \in \Lambda$, Assumption C is fulfilled.

Furthermore, notice that the iterative procedure of Theorem 4 provides better convergence rates with initial conditions closer to $\bar{\phi}$. Hence, let us pick $\bar{\phi}_0 = \bar{\phi}_{1G}$, with $\bar{\phi}_{1G}$ denoting the periodic component of the first Galerkin approximation of $\phi(t)$, namely [14]:

$$\bar{\phi}_{1G}(t) = \frac{4AB\omega(1+\lambda^2 Q)}{4+\lambda^2\omega^2 Q^2}\cos\omega t + \frac{2\lambda AB(4-\omega^2 Q)}{4+\lambda^2\omega^2 Q^2}\sin\omega t,$$

with $Q = 2A^2 + B^2$. The fact that $\bar{\phi}_{1G}(t)$ has a λ-dependent closed-form analytic expression maintains the possibility of achieving robustness by means of dynamic compensation. Finally, straightforward calculations yield $\|\bar{\phi}_{1G}\| \leq 0.8255 < L$, $\|\dot{\bar{\phi}}_{1G}\| \leq 0.5161 < D$, $\forall \lambda \in \Lambda$.

Robustness of the control approach in front of piecewise constant load disturbances achieved by means of dynamic compensation may be observed as follows. At $t = 15$ normalized time units (ntu), the output resistance is supposed to undergo the above described step change. Assuming output load measurement, a delay of 0.01 ntu between the appearance of the disturbance and the incorporation of the actual value in the inductor current reference is considered.

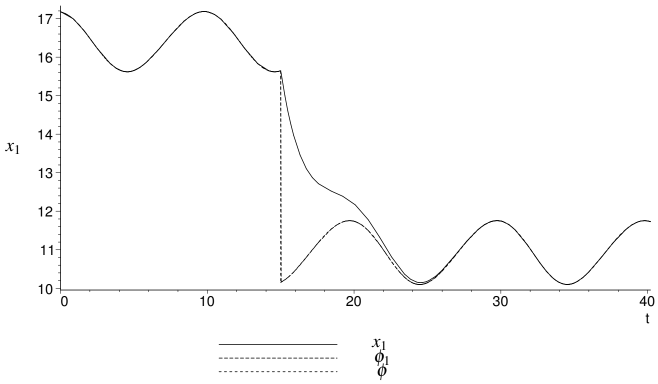

Fig. 1. The input current x_1 tracking the reference ϕ_1 under dynamic compensation of a load disturbance occurring at $t = 15$ ntu.

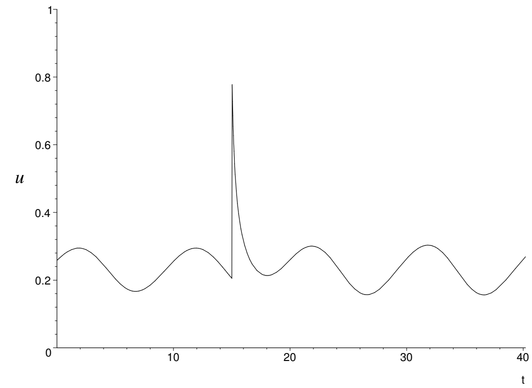

Fig. 3. The control action u accommodating a load disturbance occurring at $t = 15$ ntu.

This value is in accordance with the sampling frequency of commercially available sensing devices.

The dynamical behavior of system (3), (4) subject to the continuous state feedback control law (16), with current reference $x_{1d} = \phi_1 = g_0 + \bar{A}(\bar{\phi}_{1G})$ and $\gamma = 0.5$, has been simulated with MAPLE. Initial conditions have been set to $x_1(0) = \phi_1(0)$, $x_2(0) = x_{2d}(0)$. As indicated above, the value of λ in (4) has been changed at $t = 15$ ntu, while the updating in ϕ_1 has been carried out at $t = 15.01$ ntu.

Figure 1 depicts the input current $x_1(t)$ tracking the command profile x_{1d}. The plot also includes the exact solution of equation (8), which appears to be indistinguishable from its approximation ϕ_1. Figure 2 depicts the output voltage reference $x_{2d}(t)$ and the output voltage state variable x_2. Notice that dynamic compensation allows effectiveness of the tracking task to be recovered in a period and a half. Finally, Figure 3 shows that the control action (16) does not saturate during the entire process.

VII. CONCLUSIONS AND FUTURE WORKS

This article studies the robust tracking control problem of periodic voltage profiles in nonminimum phase DC-DC nonlinear switched power converters by means of indirect control. The procedure allows to find bounded periodic references for the unstable internal variable by providing closed-form analytic expressions uniformly convergent to the exact solution of the corresponding inverse problem. Then, piecewise constant load disturbances belonging to an a priori known compact set are rejected by means of dynamic compensation.

Further research should address the tracking of general bounded references for the output voltage, preferably causal.

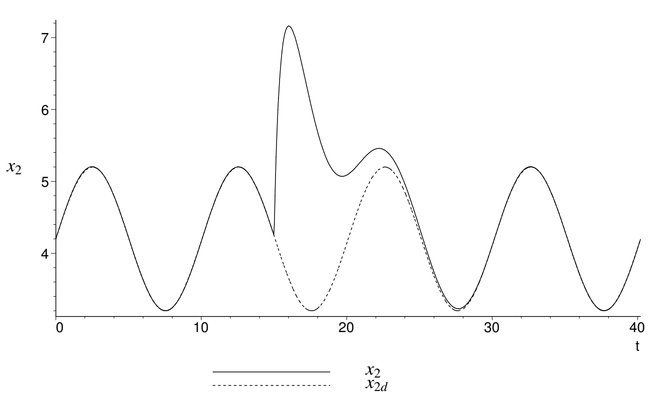

Fig. 2. The output voltage x_2 tracking the reference x_{2d} under dynamic compensation of a load disturbance occurring at $t = 15$ ntu.

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